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Axial wall conduction preheating effects in high Péclet number laminar forced convection

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Abstract—An analytical solution for the preheating due to the wall axial conduction in forced convection heating of fully developed laminar flows, is presented. The geometry of interest consists of a long circular tube heated for a finite or semi-infinite length. It is proved that in the preheating region the wall heat flux varies with exponential law in the streamwise direction, so that the temperature profile becomes fully developed. Moreover, an exact functional relation between the exponent of the wall heat flux distribution and a single parameter, which depends on the Péclet number and the wall conductance, is derived. The practical significance of the analysis is finally discussed. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

Laminar forced convection heat transfer inside ducts is usually analysed with the thermal boundary condition imposed directly to the fluid boundary, i.e. at the inside wall of the duct. Significant reviews of the most relevant literature on the subject are available, like those of Shah and London [1] and Shah and Bhatti [2]. However, in practical applications the thermal boundary condition is specified at the outer surface of the duct, while temperature and heat flux distributions at the wall–fluid interface, because of wall conduction effects, are *a priori* unknown. Since convection in the fluid and conduction in the wall are strictly connected, this problem is referred to as a conjugate problem.

In the field of conjugate heat transfer the case of ducts with a heating section preceded by a portion of unheated pipe is relevant to a variety of engineering applications. In such a situation the axial conduction in the wall can carry substantial amounts of heat upstream in the unheated region. The heat which penetrates upstream by conduction is removed from the wall by the fluid and carried downstream by convection, the two phenomena playing opposite roles. As a consequence, in this region a significant preheating of the wall and fluid occurs, anticipating the thermal development with respect to the section where the direct heating starts. If compared with the case of negligible wall heat conduction, the Nusselt number in the heating section is reduced in a significant way.

When the axial heat diffusion in the fluid and the radial temperature gradient in the wall can be considered negligible (i.e. high Péclet numbers and thin walls), the problem shows some interesting features. The analysis of the literature will be limited to this case, even if other papers on preheating effects, particularly for low Péclet numbers and turbulent flows, are available.

Hall *et al.* [3] were the first to point out that in the nondirectly heated section of the tube a situation of full thermal development is achieved. This region is then characterized by a heat flux density exchanged at the wall-to-fluid boundary varying with exponential law in the axial direction.

For thin walls, while assuming high Péclet number values, Cotton and Jackson [4] demonstrated that the conjugate effect due to wall conduction can be taken into account by means of a single dimensionless parameter, termed the conduction influence parameter, defined as

$$I = \frac{K}{Pe^2}\Delta(1+\Delta). \qquad (1)$$

In the preheating region a condition of exponential heating was numerically identified. However, their theoretical proof is not exhaustive from a mathematical stand point, because the uniqueness of the exponent was not proved. They demonstrated also that the exponential index

$$\beta = \xi R Pe \qquad (2)$$

characterizing the exponential trend in the preheating region, is determined solely by the value of I

$$\beta = \beta(I) \qquad (3)$$

and offered a numerical validation of this functional relation.

With the aim of providing a method for the measurement of the mass flow rate, Roetzel [5] proposed an analytical solution for the preheating problem. While assuming the existence of a fully developed

NOMENCLATURE

A	auxiliary function, equation (18)
c_p	constant pressure specific heat [J kg^{-1} K^{-1}]
D	inner diameter of the tube [m]
f	auxiliary function, equation (15)
F	auxiliary function, equation (A3)
${}_1F_1$	Confluent Hypergeometric Function, equation (A8)
g	auxiliary function, equation (15)
I	Conduction Influence Parameter, equation (1)
k	thermal conductivity [W m^{-1} K^{-1}]
K	wall-to-fluid thermal conductivity ratio (k_w/k_f)
l'	preheating length [m]
l^*	dimensionless preheating length (l'/RPe)
Nu	Nusselt number ($q''D/k_f(T'_w-T'_b)$)
pe	Poiseuille function, equation (A5)
Pe	Péclet number ($RePr$)
Pr	Prandtl number ($\nu\rho c_p/k$)
q''	heat flux density [W m^{-2}]
q	dimensionless heat flux (q''/q''_0)
qe	Poiseuille function, equation (A6)
r'	radial coordinate [m]
r	dimensionless radial coordinate (r'/R)
R	inner radius of the tube [m]
Re	Reynolds number (WD/ν)
s	auxiliary function, equation (A2)
T	dimensionless temperature ($(T'-T'_{in})/(q''_0R/k_f)$)
T'	temperature [K]
u'	axial velocity [m s^{-1}]
u	dimensionless axial velocity (u'/W)
W	average axial velocity [m s^{-1}]
x'	axial coordinate [m]
x^*	dimensionless axial position (x'/RPe).

Greek symbols

α	auxiliary position, equation (21)
β	exponential index (ξRPe)
Γ	Gamma function
δ	wall thickness [m]
Δ	dimensionless wall thickness (δ/D)
λ	constant of separation, equations (16) and (17)
μ	auxiliary position, equation (20)
ν	kinematic viscosity [m^2 s^{-1}]
ξ	exponent [m^{-1}]
ρ	density [kg m^{-3}]
Φ	auxiliary function, equation (24).

Subscripts

b	bulk
f	fluid
i	wall–fluid interface
in	inlet
w	wall
0	origin of axial coordinate.

situation, the asymptotic Nusselt number could be calculated analytically as a function of the measured exponent shown by the temperature distribution arising at the wall–fluid interface. Although in this analysis neither the conduction influence parameter, nor the functional relation (3) were identified, the interpolating function proposed for the calculation of the Nusselt number can be rearranged in terms of the parameter I. In this sense, the results of Cotton and Jackson [4] are confirmed, even if the existence of a fully developed thermal situation is assumed.

The preheating region has been investigated also by Pagliarini and Piva [6], in the frame of a full numerical simulation of an experimental test section, where a laminar flow was heated for a short length. For this reason the coupling with the ambient and the radial temperature gradient in the wall were also considered in the numerical simulation.

From these numerical results the following approximate correlation for the functional relation (3) was obtained by Piva and Pagliarini [7]:

$$\beta = 0.532I^{-0.610} \tag{4}$$

in the range $4.5\cdot10^{-6} < I < 5\cdot10^{-4}$ and $200 < \beta < 4000$, with an overall uncertainty of the fit equal to 2. In ref. [7], the correlation (4) was also compared with experimental data gathered in a new test rig. Based on equation (4), two original and promising methods for the measurement of thermal conductivity of ducts and mass flow rate were proposed, the latter independently from Roetzel [5].

Analysis of the literature shows how both in the numerical and experimental analyses of the conjugate preheating problem, a fully developed situation is evident [4, 6, 7], though its existence has never been completely demonstrated [3, 4] and in certain cases also assumed [5]. The available methods for the calculation of the exponential index from the conduction influence parameter are all based on interpolations of numerical or experimental results, while for the application of the promising experimental methods proposed in refs. [5, 7], an exact relation would be appreciated.

In the present paper the exact analytical expressions of the functional relation $\beta = \beta(I)$ and of the Nusselt number, $Nu = Nu(I)$, are presented. Furthermore, from the exact solution a more general proof of the existence of a full thermal development can be drawn.

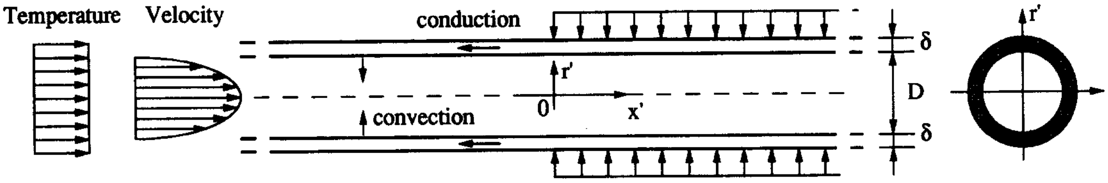

Fig. 1. Schematic representation of the conjugate preheating problem.

FORMULATION OF THE PROBLEM

The problem considered is schematically shown in Fig. 1. A Newtonian fluid in fully developed laminar flow is heated from the wall of a long circular tube. Downstream from section $x' = 0$, the outside of the wall is heated by a known distribution of heat flux density. Upstream the wall is thermally insulated. The axial heat conduction in the fluid is negligible, as is the viscous dissipation. At the inlet to the tube a uniform temperature distribution is considered both in the fluid and in the wall. The thermal properties of solid and fluid are assumed to be constant. Consideration is given to relatively thin walls so that temperature variations in the radial direction across the thickness of the wall can be neglected.

The present analysis is limited to the region upstream of the heating section.

The problem can be stated in mathematical form as follows (see the Nomenclature for the definition of the dimensionless quantities):

fluid region

$$(1-r^2)\frac{\partial T}{\partial x^*} = \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} \tag{5}$$

$$T(x^* \to -\infty) = 0 \tag{6}$$

$$\left(\frac{\partial T}{\partial r}\right)_{r=0} = 0; \tag{7}$$

solid region

$$2I\frac{\mathrm{d}^2 T_w}{\mathrm{d}x^{*2}} - q_i(x^*) = 0 \tag{8}$$

$$T_w(x^* \to -\infty) = 0; \tag{9}$$

continuity at the wall–fluid interface

$$T_w(x^*) = T(x^*, r = 1) \tag{10}$$

$$q_i(x^*) = \left(\frac{\partial T}{\partial r}\right)_{r=1}; \tag{11}$$

continuity of temperature and heat flux at $x^* = 0$

$$T(x^* = 0^-, r) = T(x^* = 0^+, r) \tag{12}$$

$$\left(\frac{\partial T}{\partial x^*}\right)_{x^*=0^-,r} = \left(\frac{\partial T}{\partial x^*}\right)_{x^*=0^+,r}. \tag{13}$$

The conditions of continuity at the wall–fluid interface allow the substitution of boundary condition (11) with

$$\left(\frac{\partial T}{\partial r}\right)_{r=1} = 2I\left(\frac{\partial^2 T}{\partial x^{*2}}\right)_{r=1}. \tag{14}$$

Equation (5) with its boundary conditions (6), (7) and (14) can be solved by separation of the variables. The solution is then written as

$$T(r, x^*) = f(r)g(x^*). \tag{15}$$

Equation (5) is then reduced to the following two ordinary differential equations:

$$\frac{\mathrm{d}g}{\mathrm{d}x^*} = \lambda g \tag{16}$$

$$\left(\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r}\right) = \lambda(1-r^2)f. \tag{17}$$

A general solution of equation (16) can be expressed in terms of the exponential function

$$g(\lambda, x^*) = A(\lambda)\exp(\lambda x^*). \tag{18}$$

Equation (17) is the so-called Poiseuille equation [9] and its solution can be expressed in terms of confluent hypergeometric functions (see Appendix A for further details). The solution of equation (17) is given by that Poiseuille function bounded at the centreline of the duct, so that boundary condition (7) is always satisfied

$$f(\lambda, r) = pe(r, \mu) = \exp(-\mu r^2/2)\,{}_1F_1[\alpha, 1, \mu r^2]. \tag{19}$$

In equation (19) the following positions are used:

$$\mu = (-\lambda)^{1/2} \tag{20}$$

$$\alpha = \frac{1}{2}\left(1 - \frac{\mu}{2}\right) \tag{21}$$

with μ and α both real or complex depending on $\lambda < 0$ or $\lambda > 0$.

The functions $f(\lambda, r)$ do not form an orthogonal set, so the solution is sought by means of the Fourier integrals method [8]. Therefore, the solution will be of the form

$$T(x^*, r) = \int_{-\infty}^{\infty} A(\lambda)\exp(\lambda x^*)f(\lambda, r)\,\mathrm{d}\lambda. \tag{22}$$

From the substitution of the general solution (22) in the boundary condition (14), follows:

$$\int_{-\infty}^{+\infty}\left[A(\lambda)\frac{\partial f(\lambda, r=1)}{\partial r}\right]\exp(\lambda x^*)\,d\lambda$$

$$= \int_{-\infty}^{+\infty}[2I\lambda^2 A(\lambda) f(\lambda, r=1)]\exp(\lambda x^*)\,d\lambda. \quad (23)$$

The admissible values of the constant of separation λ then satisfy the following transcendental equation, derived from equation (23):

$$\Phi(\lambda) = \frac{\partial f(\lambda, r=1)}{\partial r} - 2I\lambda^2 f(\lambda, r=1) = 0. \quad (24)$$

The search for the roots of this equation can be easily executed by means of a numerical procedure.

Boundary condition (9) is satisfied for positive values of λ. In this range equation (24) has got a unique root (see Appendix B for the proof). In the following, the only acceptable constant of separation λ will be indicated as β. Hence, the solution of the problem reduces to

$$T(x^*, r) = A\exp(\beta x^*) f(\beta, r). \quad (25)$$

From the solution according to equation (25) it may be easily shown that a thermally fully developed flow arises.

The temperature distribution (25) is varying axially with exponential law, as is the wall–fluid heat flux density, given by

$$q_i(x^*) = A\exp(\beta x^*)\frac{\partial f(\beta, r=1)}{\partial r}. \quad (26)$$

The integral energy balance equation, applied between $-\infty$ and x^*, gives the bulk temperature

$$T_b(x^*) = 4A\frac{\partial f(\beta, r=1)}{\partial r}\frac{\exp(\beta x^*)}{\beta}. \quad (27)$$

The Nusselt number in the preheating region, and particularly at the inlet of the heated section, is finally given by

$$Nu = \left[\frac{1}{2}\left(\frac{f(\beta, r=1)}{\frac{\partial f(\beta, r=1)}{\partial r}}\right) - \frac{2}{\beta}\right]^{-1} \quad (28)$$

independent from the axial coordinate, x^*. Equation (28) shows that the flow is thermally fully developed.

For the evaluation of equation (25), the value of the constant A must be calculated from the coupled solution of the thermal problem in the upstream and downstream regions, thus depending on the particular thermal boundary condition specified downstream. However, knowledge of the constant A is not needed for the aim of the present paper and for this reason its calculation will be avoided.

It can be observed that in the final temperature distribution, equation (25), only one constant, A, is involved; for this reason, only one boundary condition at $x^* = 0$ is enough to obtain its value. Then for the aim of the present paper, the boundary con-

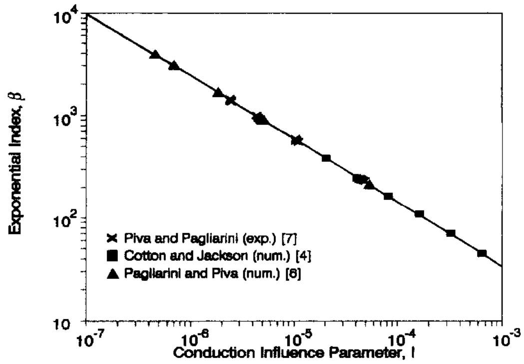

Fig. 2. Exponential index, β, as a function of the conduction influence parameter, I.

ditions (12) and (13) can also be substituted by a condition of prescribed wall temperature, or of prescribed heat flux density at $x^* = 0$.

The present solution can be used for thermal boundary conditions of assigned heat flux density distribution in the heated region, independently from the modality of heating. For thermal boundary condition of assigned temperature distribution this solution is not valid, because in this case at $x^* = 0$ both the temperature and the axial heat flux density in the wall are prescribed.

Finally, the influence of the wall axial conduction on the exponential index, β, and on the Nusselt number, Nu, is accounted for by a single dimensionless parameter, the conduction influence parameter, I, thus confirming the previous analyses [4, 6, 7].

RESULTS AND DISCUSSIONS

The problem proposed has been solved by means of a mathematical package on a personal computer. The preheating effects are discussed in terms of exponential index and Nusselt number distributions, and of preheating length.

In the following discussion, reference will be constantly made to the experimental data taken by Piva and Pagliarini [7] for the same geometry analysed in the present paper. For reason of brevity the details of the experimental arrangement and procedure are omitted here.

The root of equation (24), giving the exponential index, β, as a function of the conduction influence parameter, I, has been calculated numerically by means of the bisection method. In Fig. 2, the exponential index β is shown as a function of I. The exponential index decreases monotonically as the conduction influence parameter increases. The latter parameter takes into account both the dimensionless wall conductance, $K\Delta(1+\Delta)$ for thin walls, and the Péclet number, Pe, playing opposite roles in the process. If for fixed Péclet numbers, the wall conductance increases, the exponential index decreases and the preheating penetrates more and more upstream. This is because the wall conduction tends to overcome the

Table 1. Analytical and interpolated values of exponential index and Nusselt number

I	β an.	β equation (29)	β equation (4)	β [5]	Nu	Nu [5]
1×10^{-7}	9750.98	9752.15	9906.30	9684.67	38.33	37.81
5×10^{-7}	3648.79	3680.94	3711.42	3673.60	27.56	27.39
1×10^{-6}	2421.02	2419.47	2431.71	2417.83	23.91	23.84
5×10^{-6}	910.317	913.227	911.045	912.691	17.20	17.29
1×10^{-5}	596.401	600.263	596.914	598.879	14.94	15.07
5×10^{-5}	222.212	226.568	223.635	223.775	10.84	11.00
1×10^{-4}	144.789	148.923	146.525	145.932	9.484	9.644
5×10^{-4}	52.9318	56.1958	54.8959	53.4003	7.108	7.252
1×10^{-3}	34.0485	36.9366	35.9676	34.3426	6.373	6.505

convection from the wall to the fluid. The opposite is valid if, for fixed wall conductance, the Péclet number increases.

In Fig. 2 the experimental values taken by Piva and Pagliarini [7] and the numerical results of Cotton and Jackson [4] and Pagliarini and Piva [6] are also compared with the present analytical values, showing a very good agreement. In the light of these results, it can be concluded that, at least for the considered values of the dimensionless parameters, the secondary effects like buoyancy, finite wall thickness or axial heat diffusion in the fluid, always present in the experiments, are negligible, thus confirming the validity of the simplifying assumptions for many practical applications.

The trend shown by the exponential index in the log–log plot of Fig. 2 is almost linear, reminding of a generalized hyperbola. This trend suggests the possibility of a simplified interpolation of β as a power function of I, confirming the analysis of Piva and Pagliarini [7]. In the range analysed, the best fit, with the same weight assigned to every point, is given by:

$$\beta = 0.564 I^{-0.605} \qquad (29)$$

which is slightly different from equation (4) proposed in ref. [7], obtained from fitting a limited number of numerical results.

In order to check the validity of the interpolation, a comparison between analytical and interpolated results is reported in Table 1. In the same table the results obtained rearranging the interpolations of Roetzel [5] and those obtained with correlation (4) given in ref. [7], are also reported. The interpolating function proposed in ref. [7] gives exponential indices always slightly higher than the analytical values. Both the present interpolation, equation (29), and that of Roetzel [5] give reasonable values, the first better for low I, the latter ones for high I. It may be noted that the generalized hyperbolic interpolations give very good results only for small intervals of I; then for practical applications a proper interval of interpolation has to be chosen.

The values of the Nusselt number calculated analytically and those obtained by rearranging the equations given by Roetzel [5] in terms of the conduction influence parameter, are also reported in Table 1. The interpolating function for the Nusselt number obtained from the analysis of Roetzel [5] gives very good results, with an uncertainty in the fit within 1.5%.

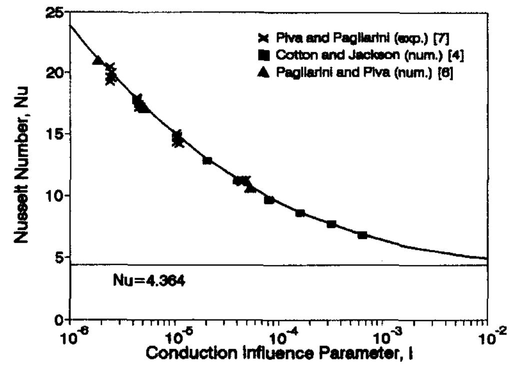

Fig. 3. Nusselt number, Nu, as a function of the conduction influence parameter, I.

The Nusselt number values in the preheating region are shown in Fig. 3 as a function of the conduction influence parameter. The trend exhibited is monotonically decreasing and for $I \to +\infty$ the $Nu = 4.364$, corresponding to a uniform heat flux density distribution, is asymptotically reached. This value is attained with good approximation just for $I = 1$ ($\beta \cong 0.125$). In Fig. 3, the experimental values taken by Piva and Pagliarini [7] and the numerical results of Cotton and Jackson [4] and Pagliarini and Piva [6] are shown for comparison, confirming the good agreement between the experimental and predicted results. If compared with the present Nusselt number values, the experimental data of ref. [7] are practically coincident over the whole range of I considered (within 2% approximation). The good agreement shown by the comparison confirms that the most important features of the phenomenon occurring in the preheating region are considered, at least in the range of validity of the present analysis, as just observed for the exponential index.

A parameter of interest for design applications is the extent of the upstream region over which the heat transferred to the fluid due to axial wall conduction is significant. This preheating length, l', is conventionally assumed to be the distance required to

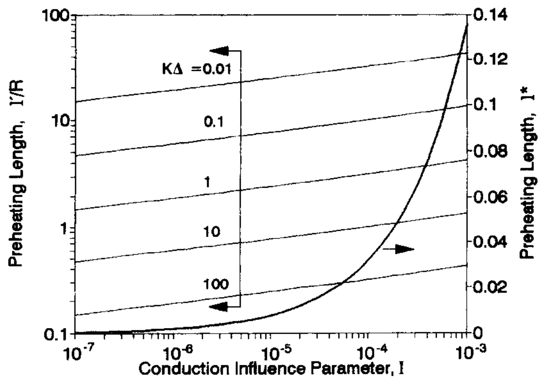

Fig. 4. Preheating length, l^*, as a function of the conduction influence parameter, I, and for fixed values of the wall conductance, $K\Delta$.

reduce the wall–fluid heat flux to 1% of the value exchanged at the beginning of the heating, when the entering fluid temperature profile is uniform. The preheating length, in dimensionless form, then becomes

$$l^* = -\frac{\ln(0.01)}{\beta}. \tag{30}$$

It is evident that, because of the functional relation $\beta = \beta(I)$, the dimensionless preheating length is a function of the conduction influence parameter only, as shown in Fig. 4. The extent of the region affected in a significant way by the heat flux redistribution increases as the parameter I increases.

For fixed characteristics of the wall, i.e. for fixed Δ and K, the preheating length decreases for increasing Péclet numbers. Then a more pronounced extent of the region affected by wall axial conduction is expected for low Péclet numbers, as shown in Fig. 4, in terms of number radii l'/R, for fixed values of the dimensionless wall conductance $K\Delta$.

CONCLUDING REMARKS

The preheating region due to wall axial conduction in circular ducts is characterized by some features of particular interest when assuming negligible axial heat diffusion in the fluid and negligible radial temperature gradients in the wall:

(1) The temperature profiles are fully developed, yielding a constant Nusselt number. Temperature and wall–fluid heat flux density increase according to an exponential law in the streamwise direction.

(2) Exponential index and Nusselt number are characterized by a single dimensionless parameter, the conduction influencer parameter, including both the wall conductance and the Péclet number.

(3) The preheating problem can be solved analytically. The solution is mathematically simple and easy to handle with small sized computers. This gives an easy way to calculate the Nusselt number at the thermal inlet of a duct in many practical situations, as a function of the conduction influence parameter only.

(4) Experimental verification of the analytical results is easily carried out; the close agreement shown between the analytical results and experimental data suggests the utilization of the configuration for the estimation of unknown parameters, like the mass flow rate flowing inside the tube or the thermal conductivity of the wall. Recent experiments [5, 7] have shown the feasibility of the proposal.

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APPENDIX A

The Poiseuille equation

$$\left(\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr}\right) - \lambda(1-r^2)f = 0 \tag{A1}$$

is a special case of the Kummer's equation [10].

When in equation (A1) we let

$$s = (-\lambda)^{1/2} r^2 = \mu r^2 \quad \text{(A2)}$$

$$F(s) = \exp(-s/2) f(\lambda, r), \quad \text{(A3)}$$

the Kummer's equation

$$sF'' + (1-s)F' - \alpha F = 0 \quad \text{(A4)}$$

can be easily obtained.

The Poiseuille equation is a homogeneous and linear second-order equation, with regular singularity at $r = 0$ and regular coefficients for all $r \neq 0$. Two independent solutions satisfy the Poiseuille equation

$$pe(r, \mu) = \exp(-\mu r^2/2)\, {}_1F_1[\alpha, 1, \mu r^2] \quad \text{(A5)}$$

and

$$qe(r, \mu) = -\left[2\Gamma(0) + \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\right] \exp(-\mu r^2/2)\, {}_1F_1[\alpha, 1, \mu r^2] \quad \text{(A6)}$$

In the present application the solution $qe(r, \mu)$ is not used because it is unbounded for $r = 0$.

The derivative of the Poiseuille function $pe(r, \mu)$ is given by

$$\frac{\partial pe(r, \mu)}{\partial r} = \mu r \exp(-\mu r^2/2)(2\, {}_1F_1'[\alpha, 1, \mu r^2] - {}_1F_1[\alpha, 1, \mu r^2]) \quad \text{(A7)}$$

For the execution of the calculations, both the confluent hypergeometric function and its derivative are needed. The confluent hypergeometric function is defined as follows [10]:

$${}_1F_1[a, b, z] = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!} \quad \text{(A8)}$$

with a, b and z real or complex numbers.

The derivative of the confluent hypergeometric function is [10]

$${}_1F_1'[a, b, z] = \frac{\mathrm{d}}{\mathrm{d}z}({}_1F_1[a, b, z]) = \frac{a}{b}\, {}_1F_1[a+1, b+1, z]. \quad \text{(A9)}$$

APPENDIX B

In this Appendix it is proved that

$$\Phi(\lambda) = \frac{\partial f(\lambda, r=1)}{\partial r} - 2I\lambda^2 f(\lambda, r=1) = 0 \quad (24)$$

has only one unique root for positive values of λ.

Equation (28) can be rewritten as

$$\frac{\partial f(\lambda, r=1)}{\partial r} = \frac{f(\lambda, r=1)}{2\left(\frac{2}{\lambda} + \frac{1}{Nu(\lambda)}\right)}. \quad \text{(A10)}$$

By introducing equation (A10) in equation (24), one obtains

$$\Phi(\lambda) = f(\lambda, r=1)\lambda\left[\frac{Nu(\lambda)}{2(\lambda + 2Nu(\lambda))} + 2I\lambda\right] = 0. \quad \text{(A11)}$$

For $\lambda = 0$, boundary condition (7) is not satisfied.

There is no root of equation

$$f(\lambda, r=1) = 0 \quad \text{(A12)}$$

for positive values of λ, as shown by Papoutsakis [11].

Therefore, the acceptable roots can be derived only from the term in square brackets. The zeros of this term are given by

$$4INu(\lambda)\lambda^2 + 8INu(\lambda)\lambda - Nu(\lambda) = 0. \quad \text{(A13)}$$

This equation is not suited for the search of the roots. Nevertheless, it can be useful to determine their sign. The roots of equation (A13) are:

$$\lambda = -Nu(\lambda)\left(1 \pm \sqrt{1 + \frac{1}{4Nu(\lambda)I}}\right). \quad \text{(A14)}$$

I is positive, as is $Nu(\lambda)$ for $\lambda > -25.6796$ [12]. Therefore, equation (A13) gives two roots, the first one positive and the second one negative. Then it is proven that only one root of equation (24) is positive and so acceptable.